

Mass Dependence of Vacuum Energy

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Abstract

The regularized vacuum energy (or energy density) of a quantum field subjected to static external conditions is shown to satisfy a certain partial differential equation with respect to two variables, the mass and the “time” (ultraviolet cutoff parameter). The equation is solved to provide integral expressions for the regularized energy (more precisely, the cylinder kernel) at positive mass in terms of that for zero mass. Alternatively, for fixed positive mass all coefficients in the short-time asymptotics of the regularized energy can be obtained recursively from the first nontrivial coefficient, which is the renormalized vacuum energy.

BACKGROUND

The experimentally documented physical relevance of vacuum energy involves the electromagnetic field (the Casimir effect). Other theoretical work also usually considers massless fields (scalar fields for calculational and conceptual simplicity; neutrinos or gluons for more exotic or speculative applications). Nevertheless, the vacuum energy of a quantum field with mass is of theoretical importance. Adding the mass term is the simplest generalization of the basic model of a massless scalar field in an empty flat cavity, a step toward the more complicated scenarios with space-dependent external potentials or curved background geometries. Although massive Casimir-type effects in the simplest geometries tend to be exponentially damped relative to their massless counterparts, they can be less negligible in other geometries [1, Secs. 7.4–7.5].

The massive field in one-dimensional space was studied by Hays [2]. (See also [3, 4, 5].) Perhaps the most notable feature of the results is the presence of a logarithmic divergence, absent from the massless case. Although physically harmless in the context of the one-dimensional bag model [2, 3], a logarithmic divergence is regarded as particularly problematic by some theorists because it is not automatically eliminated by dimensional or zeta-function regularization.

As I have stressed elsewhere [6, 7, 8], vacuum energy is one of a series of moments of the spectral distribution of the differential operator appearing in the field equation; these quantities arise as the coefficients in the short-time asymptotics of a certain Green function, the “cylinder kernel”. The diagonal value of the latter is, up to a differentiation and a numerical factor, the regularized vacuum energy defined by an ultraviolet cutoff (with the time variable as the cutoff parameter) [2, 3, 9]. Thus the vacuum energy (regardless of experimental relevance) is of mathematical interest as a tool of spectral analysis. The cylinder-kernel coefficients incorporate nonlocal geometrical information that is not extractible from the much-studied short-time asymptotic expansion of the heat kernel. (The cylinder kernel is minus twice the t -derivative of the Green function employed in [2, 3].)

NOTATION

Consider a field equation of the type $\frac{\partial^2 \phi}{\partial t^2} = -H\phi$, where

$$H = H_0 + \mu, \quad \mu \equiv m^2, \quad (1)$$

and H_0 is a self-adjoint second-order differential operator in the spatial variables, such as $-\nabla^2$. For simplicity (although the treatment of local energy density is actually more general) assume that the spatial domain is compact and H has a discrete, positive spectrum $\{\omega_n^2\}$ with orthonormal eigenfunctions $\{\phi_n(x)\}$.

For auxiliary mathematical purposes one studies the *heat kernel*

$$K(t, x, y) = \langle x | e^{-tH} | y \rangle = \sum_{n=1}^{\infty} e^{-t\omega_n^2} \phi_n(x) \phi_n(y)^* \quad (2)$$

and the *cylinder (Poisson) kernel*

$$T(t, x, y) = \langle x | e^{-t\sqrt{H}} | y \rangle = \sum_{n=1}^{\infty} e^{-t\omega_n} \phi_n(x) \phi_n(y)^*. \quad (3)$$

Each of these can be “traced” over space; for example,

$$\mathrm{Tr} T = \int \langle x | e^{-t\sqrt{H}} | x \rangle dx = \sum_{n=1}^{\infty} e^{-t\omega_n}. \quad (4)$$

Formally, the vacuum energy of the quantized field configuration is

$$E = \frac{1}{2} \sum_{n=1}^{\infty} \omega_n = -\frac{1}{2} \lim_{t \rightarrow 0} \frac{\partial}{\partial t} \mathrm{Tr} T, \quad (5)$$

and one possible definition (see [7, 8]) of the vacuum energy density is

$$T_{00}(x) = \frac{1}{2} \sum_{n=1}^{\infty} \omega_n \phi_n(x) \phi_n(y)^* = -\frac{1}{2} \lim_{t \rightarrow 0} \frac{\partial}{\partial t} T(t, x, x). \quad (6)$$

In reality, the limits in (5) and (6) do not exist, but $\mathrm{Tr} T$ and $T(t, x, x)$ possess asymptotic expansions as $t \downarrow 0$ of the form [6, 9, 10, 11]

$$T \sim \sum_{s=0}^{\infty} e_s t^{-d+s} + \sum_{\substack{s=d+1 \\ s-d \text{ odd}}}^{\infty} f_s t^{-d+s} \ln t, \quad (7)$$

where the coefficients of the divergent terms are simple, local objects that can be absorbed by renormalization. (Here d is the spatial dimension.) Therefore, one regards (3) and (4), after operation by $-\frac{1}{2} \frac{\partial}{\partial t}$, as the *regularized* energy and energy density, and one regards $-\frac{1}{2}$ times the coefficient of the term of order t in (3) and (4) as the *renormalized* energy and energy density:

$$E \text{ or } T_{00} = -\frac{1}{2} e_{d+1}. \quad (8)$$

Similarly, if K stands for either $\mathrm{Tr} K$ or $K(t, x, x)$, it has an expansion of the form

$$K \sim \sum_{s=0}^{\infty} b_s t^{(-d+s)/2}. \quad (9)$$

THE MAIN EQUATION

The coefficients in (7) and (9) are functions of μ . Let us write $T(\mu, t)$ and $K(\mu, t)$ for the quantities being expanded and write $e_s(\mu)$, etc., for the coefficients. In the case of the heat kernel, it is elementary that

$$K(\mu, t) = K(0, t) e^{-\mu t}, \quad (10)$$

and from (10) it is routine to find formulas for the $b_s(\mu)$ in terms of $b_{s'}(0)$ ($s' \leq s$). For the cylinder kernel it is clear that no elementary factorization like (10) occurs, and hence the mass dependence is much more interesting and nontrivial.

On the other hand, (10) is equivalent to the differential equation

$$\frac{\partial K}{\partial \mu} = -tK. \quad (11)$$

The goal of the present paper is to find, as nearly as possible, an analogue of (11) for the quantities $T(\mu, t)$ related to vacuum energy. Since $\omega_n(\mu) = \sqrt{\omega_n(0)^2 + \mu}$, it is easy to show from (4) or (3) that

$$\frac{\partial^2}{\partial \mu \partial t} \left(\frac{T}{t} \right) = \frac{T}{2}, \quad (12)$$

which is the central equation of this paper. The variables t and μ naturally range from 0 to $+\infty$.

If its right side were zero, (12) would be mathematically equivalent to the massless wave equation in two-dimensional space-time written in null (light cone) coordinates; as is well known, its general solution would then be $T(\mu, t)/t = A(t) + B(\mu)$, where A and B are arbitrary functions. The full equation (12) is of the same hyperbolic type, and one can again expect the general solution to involve two arbitrary one-variable functions. One of these should be the “initial value” $T(0, t)$, in analogy with (10). The remaining boundary condition is (cf. (3)–(4))

$$\lim_{t \rightarrow +\infty} T(\mu, t) = 0. \quad (13)$$

SOLUTION BY LAPLACE TRANSFORM

Let $F(s, t)$ be the Laplace transform of $T(\mu, t)/t$ with respect to μ . Then (12) is equivalent to

$$s \frac{dF}{dt} - \frac{\partial}{\partial t} \frac{T(0, t)}{t} = \frac{t}{2} F;$$

i.e.,

$$\frac{dF}{dt} - \frac{t}{2s} F = \frac{\partial}{\partial t} \frac{T(0, t)}{st}. \quad (14)$$

The solution of (14) consistent with (13) is

$$F(s, t) = -e^{t^2/4s} \int_t^\infty e^{-v^2/4s} \frac{\partial}{\partial v} \frac{T(0, v)}{sv} dv. \quad (15)$$

Equivalently,

$$F(s, t) = \frac{T(0, t)}{st} - \frac{1}{2s^2} e^{t^2/4s} \int_t^\infty e^{-v^2/4s} T(0, v) dv. \quad (16)$$

Thus, in principle, $T(\mu, t)$ can be calculated from $T(0, v)$.

Indeed, the inverse Laplace transform can be performed at the kernel level (under the integral sign in (15) or (16)) [12, p. 1026]:

$$\frac{T(\mu, t)}{t} = - \int_t^\infty J_0(m\sqrt{v^2 - t^2}) \frac{\partial}{\partial v} \left(\frac{T(0, v)}{v} \right) dv \quad (17)$$

or

$$T(\mu, t) = T(0, t) - t \int_t^\infty \frac{m dv}{\sqrt{v^2 - t^2}} J_1(m\sqrt{v^2 - t^2}) T(0, v). \quad (18)$$

A change of variable ($w^2 = v^2 - t^2$) converts (18) to

$$T(\mu, t) = T(0, t) - t \int_0^\infty \frac{m dw}{\sqrt{w^2 + t^2}} J_1(mw) T(0, \sqrt{w^2 + t^2}), \quad (19)$$

and there is a similar variant of (17).

Example 1

The cylinder kernel of the free massless scalar field in spatial dimension d is

$$T(0, t, \mathbf{x}, \mathbf{y}) = \frac{\Gamma(c)\pi^{-c}t}{(t^2 + z^2)^c} \quad (20)$$

where $z \equiv |\mathbf{x} - \mathbf{y}|$ is the spatial separation and $c = \frac{1}{2}(d + 1)$. According to (17), therefore,

$$T(\mu, t, \mathbf{x}, \mathbf{y}) = 2c\Gamma(c)\pi^{-c}t \int_0^\infty \frac{wJ_0(mw)dw}{(w^2 + t^2 + z^2)^{c+1}}. \quad (21)$$

From [14, p. 425] follows

$$T(m^2, t, \mathbf{x}, \mathbf{y}) = 2^{1-c}\pi^{-c}m^c t(t^2 + z^2)^{-c/2} K_c(m\sqrt{t^2 + z^2}), \quad (22)$$

which is the correct formula¹ for the cylinder kernel of the free field of mass m . All the solution formulas for problems with infinite flat boundaries now follow by the method of images.

Example 2

Let $d = 1$ and consider an interval of length L with Dirichlet boundary conditions. For the massless case the solution by images can be summed in closed form [6, (23) and (27)] with the result

$$\text{Tr } T(0, t) = \frac{1}{2} \frac{\sinh(\pi t/L)}{\cosh(\pi t/L) - 1} - \frac{1}{2}. \quad (23)$$

We apply (19) and (5) and compare with the conclusions of Hays [2] about the renormalized total energy in the massive case (finding complete agreement). It is convenient to separate out the contribution of the free Green function (20) (with $c = 1$, $z = 0$, and integrated over $0 < x < L$), because that is where all the divergences lie. That is, in both places in (19) write

$$T(0, v) = \frac{L}{\pi v} + \left[T(0, v) - \frac{L}{\pi v} \right].$$

When we apply the operator $-\frac{1}{2} \lim_{t \rightarrow 0} \frac{\partial}{\partial t}$ to (19) we thus encounter four terms:

- The divergent term

$$\frac{L}{2\pi t^2} \quad (24)$$

(present already in empty space) is the mass-independent part of the renormalization of the bag constant in [2, (3.10)].

¹ The Bender–Hays Green function [2, 3] is the Green function of the Helmholtz equation in one higher dimension, $[-\nabla^2 - \frac{\partial^2}{\partial t^2} + m^2]G = \delta(t)\delta^d(\mathbf{z})$. In the free case this kernel is known [15, (4.25)] and is proportional to $K_{c-1}(m\sqrt{t^2 + z^2})$. Differentiation and a recursion relation for the modified Bessel function then lead to (22).

- The remaining (bracket) contribution of the first term in (19) is the familiar massless Casimir energy,

$$-\frac{\pi}{24L}. \quad (25)$$

(It comes from the $O(t^2)$ term in the Taylor expansion of (23). Regrettably, that crucial term is written in [6, (27)] with the wrong sign.)

- The contribution of the free Green function to the integral in (19) works out to

$$\frac{m^2 L t}{2\pi} \ln\left(\frac{mt}{2}\right) + (2C - 1) \frac{m^2 L t}{4\pi}$$

($C = 0.577\dots$, Euler's constant). The corresponding term in the regularized energy,

$$-\frac{m^2 L}{4\pi} \ln\left(\frac{mt}{2}\right) - (2C + 1) \frac{m^2 L}{8\pi}, \quad (26)$$

is the mass-dependent part of the renormalization [2, (3.10)]. It, also, is present in empty space. (It includes a finite term, proportional to $m^2 L$, which is actually ambiguous in the sense that the scale factor in the argument of the logarithm function is arbitrary.)

- The remaining (bracket) part of the integral splits into two disparate pieces.

- The term $-\frac{1}{2}$ in (23) contributes²

$$-\frac{1}{4} \lim_{t \rightarrow 0} \frac{\partial}{\partial t} (1 - e^{-mt}) = -\frac{m}{4} \quad (27)$$

to the energy, in agreement with [2, (3.15)]. This constant term, associated with paths that reflect from the boundaries an odd number of times [2, Sec. 4], represents the energy of interaction of the massive field with the two boundaries separately (i.e, it survives when L approaches infinity, and it does not contribute to the Casimir force).

- What remains is the contribution of the paths that reflect an even number of times; it is the mass-dependent part of the true Casimir energy. In our present approach it equals³

$$\begin{aligned} & \frac{m}{4} \int_0^\infty \frac{J_1(mw)}{w} \left[\frac{\sinh(\pi w/L)}{\cosh(\pi w/L) - 1} - \frac{2L}{\pi w} \right] dw \\ &= \frac{m}{4} \int_0^\infty J_1\left(\frac{mLu}{\pi}\right) \left[\coth\left(\frac{u}{2}\right) - \frac{2}{u} \right] \frac{du}{u}. \end{aligned} \quad (28)$$

In Hays's approach [2, (3.11) and (3.13)] it appears as

$$-\frac{m}{2\pi} \sum_{s=1}^\infty \frac{K_1(2mLs)}{s} + \frac{\pi}{24L} \quad (29)$$

² This follows from [13, (6.552.1)] and the power series of the Bessel functions.

³ The error made by setting $t = 0$ inside the integrand of this term of (19) before differentiating is of order $t^2 \ln t$, so it vanishes in the limit.

(since Hays's sum includes the massless Casimir energy, (25)). The equivalence of (28) and (29) is not obvious, but it can be verified by the method of [14, p. 427]. (It has also been tested numerically.) We claim no practical advantage for the integral, since the sum converges faster.

PARTIAL SOLUTION BY RECURSION

Rarely will one of the integrals (15)–(19) be evaluable analytically in any particular case. Moreover, $T(0, t)$ often will not be available for arbitrarily large t . It is worthwhile, therefore, to see how much information can be obtained from the known asymptotic structure (7) if the coefficients for $m = 0$ are known.

By substituting (7) into (12) one obtains

$$\begin{aligned} \sum_{s=0}^{\infty} (-d + s - 1) \frac{\partial e_s}{\partial \mu} t^{-d+s-2} + \sum_{\substack{s=d+1 \\ s-d \text{ odd}}}^{\infty} (-d + s - 1) \frac{\partial f_s}{\partial \mu} t^{-d+s-2} \ln t + \sum_{\substack{s=d+1 \\ s-d \text{ odd}}}^{\infty} \frac{\partial f_s}{\partial \mu} t^{-d+s-2} \\ = \sum_{s=2}^{\infty} \frac{e_{s-2}}{2} t^{-d+s-2} + \sum_{\substack{s=d+3 \\ s-d \text{ odd}}}^{\infty} \frac{f_{s-2}}{2} t^{-d+s-2} \ln t. \end{aligned} \quad (30)$$

Therefore, we have the recursion relations

$$(-d + s - 1) \frac{\partial f_s}{\partial \mu} = \frac{f_{s-2}}{2} \quad (31)$$

for $s - d$ odd and positive, and

$$(-d + s - 1) \frac{\partial e_s}{\partial \mu} = \frac{e_{s-2}}{2} - \frac{\partial f_s}{\partial \mu} \quad (32)$$

for all nonnegative integers s , the terms being set to 0 when not defined. Generically these equations can be solved recursively for $f_s(\mu)$ and $e_s(\mu)$, respectively (the initial data $f_s(0)$ and $e_s(0)$ being presumed known). Exceptions occur when $-d + s - 1 = 0$ (i.e., $s = d + 1$); then (31) becomes a tautology and (32),

$$\frac{\partial f_{d+1}}{\partial \mu} = \frac{e_{d-1}}{2}, \quad (33)$$

takes its place as the equation determining f_{d+1} .

Thus there is no equation to determine $e_{d+1}(\mu)$. The reason is that there is no way in this approach to impose the second boundary condition (13), so the solution must involve an arbitrary function. Ironically, that function turns out to be naturally identified with the renormalized vacuum energy (8), precisely the quantity of greatest physical interest. In fact, the only coefficients that have been completely determined by this exercise are the ones that are equivalent to heat-kernel coefficients [6, 7, 8]. Nevertheless, the calculation clarifies the structure of the problem and shows that once $e_{d+1}(\mu)$ is known (along with the mass-zero coefficients), all the higher cylinder-kernel coefficients are computable.

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